Introduction to Modular Arithmetic

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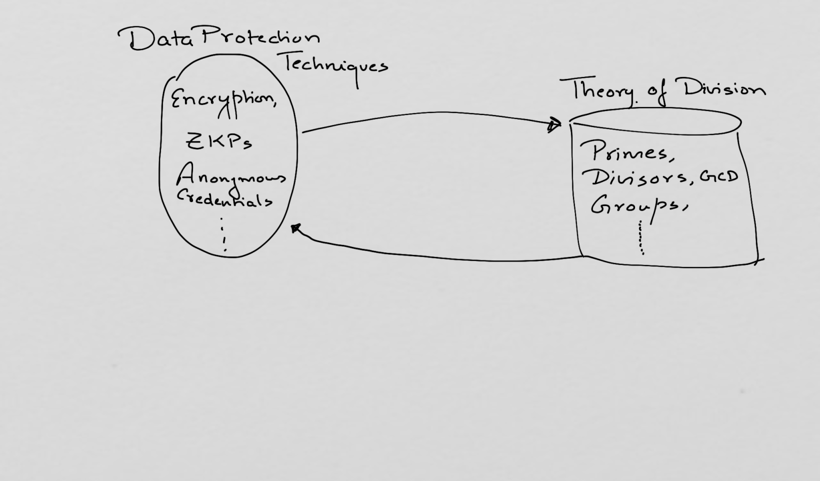
Detour


Data Protection Techniques
Encryption,
ZKPs
Anonymous Credentials
Theory of Division
Primes,
Divisors, GCD
Groups,

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    - Example: $[3]_7 = \{\cdots, -11, -4, 3, 10, 17, \cdots\} = [-4]_7 = [10]_7$
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  - Thus, $a \in [b]_n$ is the same as writing $a \equiv b \; (mod \; n)$.
- Set of all such equivalence classes: $\mathbb{Z}_n = \{[a]_n : 0 \leq a \leq n-1\}$ will be read as $\{0, 1, \cdots, n-1\}$

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  - Relatively prime integers: $gcd(a,b) = 1$
- Note: No efficient solution for integer factorization.

Euclid's Greatest Common Divisor Algorithm

- Euclid in his "The Elements" (c. 300 BC) gave a recursive algorithm: $gcd(a, b) = gcd(b, a \ mod \ b)$
  - Let $d = gcd(a, b)$. Then $d \mid a, d \mid b$.
  - $a \ mod \ b = a - qb$ where $q = \lfloor a/b \rfloor$. Thus, $d \mid a \ mod \ b$
  - Similarly, can be shown that $a \ mod \ b \mid d$
- Eg:

$$
\begin{aligned}
gcd(30, 21) &= gcd(21, 9) \\
&= gcd(9, 3) \\
&= gcd(3, 0)
\end{aligned}
$$

Extended Euclid's Algorithm

$d = gcd(a, b) = ax + by$.

- The algorithm solves for x and y. Note that x and y can be zero or negative.
- As efficient as $gcd(a, b)$ computation
- Required to compute modular multiplicative inverses.

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- Observe: $|\mathbb{Z}_n^*| < |\mathbb{Z}_n|$ when n is composite. Why?
- In practice, we choose $\mathbb{Z}_p^*$ where p is prime.

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- Eg: $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. Choose $a = 2$. Then $a^{(1)} = 2, a^{(2)} = 4, a^{(3)} = 0, \cdots$ (since $\oplus = +$). For $\mathbb{Z}_6$, we have:
  $\langle 1 \rangle = \{0, 1, 2, 3, 4, 5\}$
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Modular Arithmetic (Continued): Subgroups

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- If $(S, \oplus)$ is a finite group with identity e, then for all $a \in S$: $a^{(|S|)} = e$
  - Proof from *Lagrange's Theorem* that $ord(a) \mid |S|$

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- Thus, the above equation has a solution if and only if $[b] \in \langle a \rangle$.
  - Precise characterisation: $\langle a \rangle = \langle d \rangle = \{0, d, 2d, \cdots, (n/d - 1)d\}$, where $d = gcd(a, n)$. Thus, $|\langle a \rangle| = n/d$.
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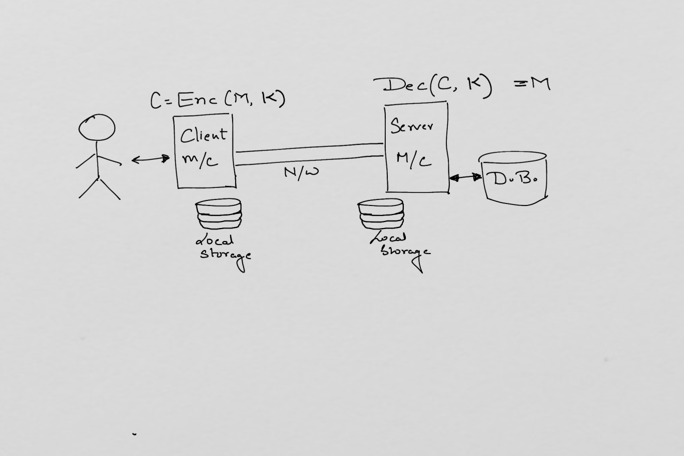
- when $d = 1 \Rightarrow$ the above equation has a *unique* solution.
- Of special interest: $b = 1$ (*multiplicative inverse* of a)

Symmetric Key Encryption



- The same key k is used for Encryption and decryption key
- Encryption produces ciphertext $C = E(M, k)$. Decryption recovers the message $M = D(E(M, k), k)$

Symmetric Key Encryption (Continued)

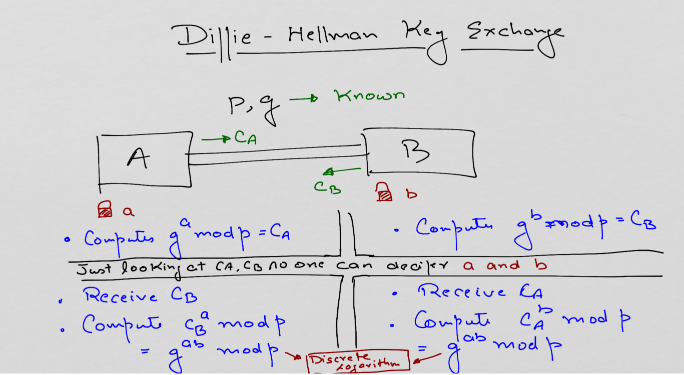
- Substitution ciphers as encryption functions: Cipher alphabet shifted, reversed, or scrambled (Eg: Caesar cipher)
  - MEETME $\rightarrow$ LOOQ LO
  - Security is weak: Frequency distribution of ciphertext which can allow formation of partial words. O is used 3 times. In English, top letters that are frequent used are E, T, A etc. Replacing O with E gives a partial word.
- Similarly, for Transposition cipher: Sliding alphabet of ciphertexts to look for anagrams. Then search the space of anagrams.
- Need to rely on a key whose detection is hard - prime factorisation of large semi-primes is presumably hard!
- Known Symmetric encryption algorithms: AES, 3DES, Blowfish.
- AES128: Runs in 16 rounds. Each round has substitution, permutation, linear transformation, XOR with round key.

More on Symmetric Encryption

- How to securely share the secret key among each pair of communicating parties?
  - Solution: Diffie-Hellman key exdchange protocol.
- After receiving the secret key, how to securely *store* them? Threats from *insider attacks*, compromised privileged software (such as the OS).
  - Note that no data protection technique via key-based data encryption will be adequate without a solution to the secure key storage problem.
- The number of keys to be maintained by each machine is $O(n)$ (where n is the number of machines that it will communicate).

Diffie-Hellman Key Exchange Protocol

- Security of the protocol is derived from the presumed hardness of the *discrete logarithm* problem.
- Protocol begins by choosing a publicly agreed upon a large prime p and the associated primitive root g.
  - Recall that primitive root is that special element $g \in \mathbb{Z}_p^*$ such that $\langle g \rangle = \mathbb{Z}_p^*$.
- Two participants A and B, then choose secret keys a and b, respectively.



Diffie-Hellman Key Exchange Protocol

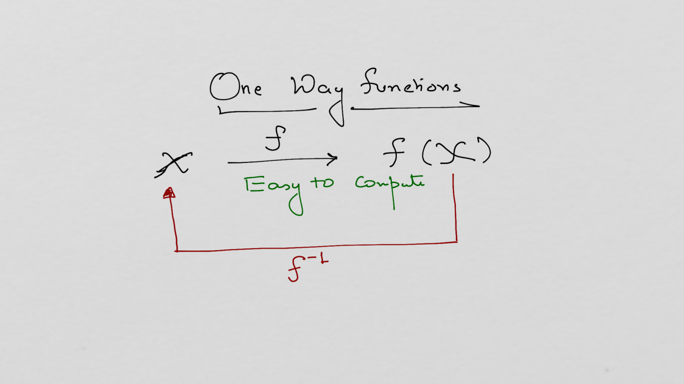
- Participant A computes a ciphertext $C_A = g^a \ (mod\ p)$. Similarly, B computes $C_B = g^b \ (mod\ p)$.
- Participant A sends C_A to participant B and receives C_B from B.
- A computes $C_B^a \ (mod\ p) = g^{ab} \ (mod\ p)$ and B computes $C_A^b \ (mod\ p) = g^{ab} \ (mod\ p)$.
- Thus, the secret key $g^{ab} \ (mod\ p)$ is established.
- Any intruder wishing to read the message will have to find the value ab (*i.e.*, solving the discrete logarithm problem).

Discrete Logarithm Problem

Let us focus on $\mathbb{Z}_n^*$ instead of $\mathbb{Z}_n$.

- We know that for all $a \in \mathbb{Z}_n^*$, $a^{|\mathbb{Z}_n^*|} \equiv 1 \ (mod\ n), n > 1$
  - This is also called *Euler's Theorem*
  - The Euler Phi function is defined as: $\phi(n) = |\mathbb{Z}_n^*|$
- Remember from earlier discussion that $|\mathbb{Z}_p^*| = p - 1$ when p is a prime.
- From Fermat's Theorem: $a^{p-1} \equiv 1 \ (mod\ p)$ for all $a \in \mathbb{Z}_p^*$
- Let $g \in \mathbb{Z}_n^*$ such that $\langle g \rangle = \mathbb{Z}_n^*$. Then $\mathbb{Z}_n^*$ is called *cyclic*.
- By definition of $\langle g \rangle$, for all $a \in \mathbb{Z}_n^*$, there exists z s.t. $g^z \equiv a \ (mod\ n)$.
  - z is called the *discrete logarithm* of a modulo n.

One Way Functions



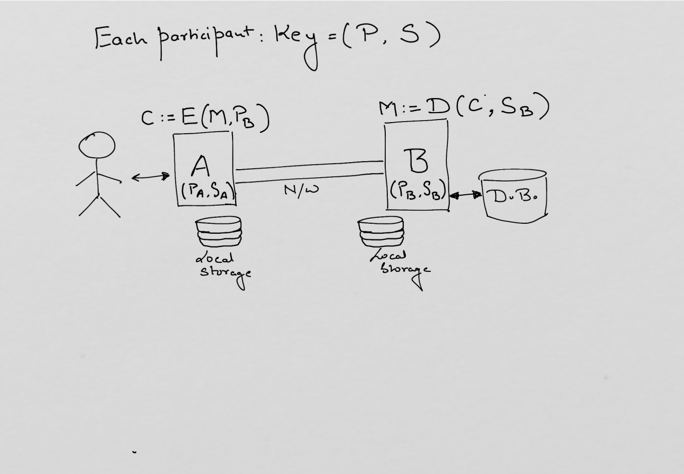
- Given x, computing $F(x)$ is fast.
- However, given $F(x)$, computing $F^{-1}(x)$ is difficult
- Discrete logarithm problem is an instance of a one-way function! That is given g, z, n computing $g^z(mod\ n)$ is fast. But given g, n, a computing $log_g(a)$

One-way Hash Functions

$F(\langle$msg-arbitrary-size$\rangle) = \langle$msg-fixed-size$\rangle$

- Properties:
  - Deterministic: same message produces the same hash.
  - Collision-resistant: It is hard to find two inputs m_1, m_2 s.t. $m_1 \neq m_2$ but $F(m_1) = F(m_2)$.
  - Avalanche effect: A small change in message leads to a large change in the hashed message
- Used in digital signatures, MACs. Egs: SHA-256, MD5
- Security: Brute-force search, Caching the o/p of hash functions (called rainbow table attack).
  - Use of *salt* (a random data as an additional input to the hash function) makes the attack infeasible.

Public-key Cryptosystems



- Every participant computes and maintains a key.
- Each key has two parts: public P, secret S

Public-key Cryptosystems(Cont.)

- Thus, machine A's key is (P_A, S_A) and B's key is (P_B, S_B).
- With a slight abuse of notation we will consider $E(M, P_x)$ in the figure as $P_x(M)$ and $D(M, S_x)$ as $S_x(M)$.
- Public and secret keys are "matched pairs", in the sense that they specify functions that are inverses of each other, *i.e.*, $S_x(P_x(msg)) = P_x(S_x(msg))$.
- Security assumption: Even though P_x is known publicly for all x, it is *hard* for an intruder to ascertain S_x from P_x. Only the owner x can compute S_x in a practical amount of time.
- Data Confidentiality: Assume A is the sender and B is the recipient of a message M. Then A encrypts by applying P_B of B, *i.e.* $C = P_B(M)$, Thus, only B can decode this message with S_B (*i.e.*, $S_B(P_B(M)) = M$)
- Digital signatures can also be implemented with Public-key cryptosystems: A can send a message M by encrypting it as $S_A(M)$. Note that any machine with P_A can decrypt this message. However, only A could have sent this message, since S_A is a secret known only to A.

Public-key Cryptosystems: RSA

A popular public-key cryptosystem is the Rivest–Shamir–Adleman algorithnm (authors given Turing Award in 2002)

1. Select two very large primes p and q [Use the probabilistic Miller-Rabin or Solovay-Strassen]
2. Compute $n = pq$. Compute $\phi(n) = (p-1)(q-1)$.
3. Choose an odd e s.t. $1 < e < \phi(n)$ and $gcd(e, \phi(n)) = 1$ [Use Euclid's gcd computation to select e]
4. Compute d as the multiplicative inverse of e, modulo $\phi(n)$. That is $ed \equiv 1 \ (mod\ \phi(n))$ [Apply Extended Euclid to solve for x s.t. $gcd(e, \phi(n)) = 1 = ex + \phi(n)y$]
5. Publish the public key $P = (e, n)$ of the participant
6. Publish the private key $S = (d, n)$ of the participant
7. The domain of a message $\mathcal{D}$ is $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$.
8. Thus $P(M) = M^e \ (mod\ n) = C$. And $S(C) = C^d \ (mod\ n)$.

Why does RSA work?

- Note $P(S(M)) = S(P(M)) = M^{ed}\ (mod\ n)$.
- Also, $ed = 1 + k(p-1)(q-1)$
- So
  $M^{ed}\ (mod\ p) = M(M^{p-1})^{k(q-1)}\ (mod\ p) = M(1)^{k(q-1)}\ (mod\ p)$
  [Follows from Fermat's Theorem]
- Repeating the same argument, we will get
  $M^{ed}\ (mod\ q) = M\ (mod\ q)$. For all M

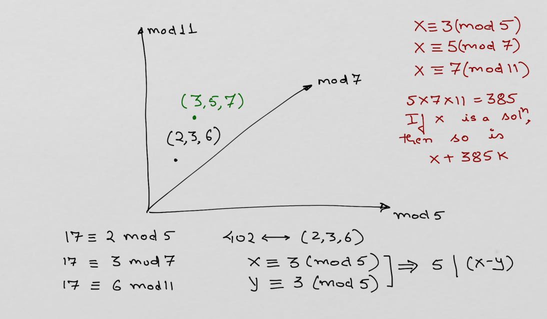
$$M^{ed} \equiv M\ (mod\ p)$$
$$M^{ed} \equiv M\ (mod\ q)$$

- From Chinese remainder theorem, $M^{ed} \equiv M\ (mod\ n)$

Chinese Remainder Theorem

If $p_1, p_2, \cdots, p_k$ are pairwise relatively prime, then for any integers $a_1, a_2, \cdots, a_k$, the set of equations: $x \equiv a_i \ mod \ p_i$ has a unique solution modulo $p_1 p_2 \cdots p_k$.

- Eg: $x \equiv 3 \ mod \ 5$
  $x \equiv 5 \ mod \ 7$
  $x \equiv 7 \ mod \ 11$



Chinese Remainder Theorem (Cont.)

- Consider three numbers $x_1, x_2, x3$ corresponding to the coordinates $(1,0,0), (0,1,0), (0,0,1)$ respectively.
- Then the point corresponding to the point $(3,5,7)$ is $3x_1 + 5x_2 + 7x_3$.
- For x_1:

$$x_1 \equiv 1 \ (mod\ 5) \quad (1)$$
$$x_1 \equiv 0 \ (mod\ 7) \quad (2)$$
$$x_1 \equiv 0 \ (mod\ 11) \quad (3)$$

- $7 * 11 \mid x_1$. Thus $77x_1' \equiv 1 \ (mod\ 5)$. Using eqn (1), we get $x_1 = 231$.
- Similarly, one can compute $x_2 = 330$ and $x_3 = 210$.
- Thus, $3x_1 + 5x_2 + 7x_3 = 3813$. Take factors of 385 out. The smallest positive number left is: 348 (solution to the original set of modular linear equations).

Chinese Remainder Theorem (Cont.)

- Provides a correspondence between a system of equations modulo a set of pairwise relative prime and an equation modulo the product of those pairwise relative primes
- "Structure Theorem" – describes the structure of $\mathbb{Z}_n$ is identical to that of $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$.
- As a result: Design of efficient algorithms (since working with $\mathbb{Z}_{n_i}$ is more efficient that working with $\mathbb{Z}_n$.

Security and Runtime Complexity of RSA

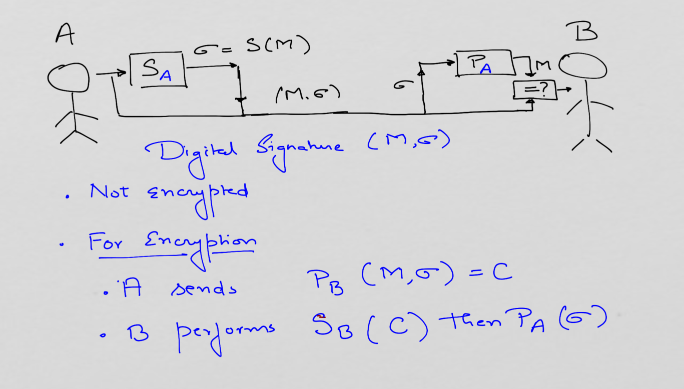
Security of RSA

- $M^{ed} \equiv M \ (mod \ n)$. To derive e and d, one will have to factor n. Typically, n is a product of two 1024 bit (300 digit) primes.

Runtime Complexity

- Applying P requires $O(1)$ modular multiplications. Applying S requires $O(\beta)$ modular multiplications (where β is the number of bits used to represent n).

Digitial Signatures



- A's digital signature for message M: $(M, S_A(M))$
- B upon receiving the signature decrypts $P_A(S_A(M))$ and performs the check $P_A(S_A(M)) \stackrel{?}{=} M$

Digital Signatures (continued)

- Note however, that the message M is sent over as plaintext
- An efficient approach is to combine data encryption with *Cryptographic hash functions*.
- CHF: allow fixed-length message fingerprints (provides *message integrity*
- A's digital signature for the message M: $\sigma = S_A(h(m))$. A sends the message $C = P_B(M, \sigma)$.
- Now, no eavesdropper can get the message in plaintext.
- Upon receiving the ciphertext, B decrypts by performing $S_B(C)$ and extracts the message: $(M, S_A(h(M)))$. It further performs the check $h(m) \stackrel{?}{=} P_A(S_A(h(m)))$.

Digital Certificates

- Certificates makes distributing public keys much easier
- An actor A can obtain a signed message from a publicly trusted authority T stating: A's public key is P_A.
- Actor A can include this certificate in her signed message.
- The recipient can now verify her signature with A's public key and the certificate from T.
- The recipient can now trust that A's key is indeed hers because of public trust in T.