The Fibonacci Sequence: A Comprehensive Review

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Abstract: This paper provides a comprehensive survey of the Fibonacci sequence, exploring its fundamental properties, various mathematical applications, and its extensions into related sequences. We begin by examining the classical Fibonacci sequence, defined by the recurrence relation $F(n) = F(n-1) + F(n-2)$ with initial conditions $F(0) = 0$ and $F(1) = 1$. The sequence's inherent characteristics, such as its mathematical identities, its connection to the golden ratio, Pascal's triangle, and its combinatorial interpretations, are thoroughly analyzed. Additionally, we investigate its extension to negative indices, known as negafibonacci numbers, and its generalization in matrix form. Through this survey, we aim to highlight the versatility and profound implications of Fibonacci-like sequences in various fields of mathematics, science, and art, showcasing their enduring significance and diverse applications.[1]

[1]The authors completed this exposition as part of the summer 2020 PROMYS program. We thank the PROMYS program for their support and for creating such a stimulating work environment. We also thank Professor David Fried and Arpoitri Ghosal for their helpful feedback. The project contains no references because it was undertaken without consulting any online resources; all results were developed from scratch by the authors.

Contents

1 Introduction

The first few Fibonacci numbers are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...$$

where each term in the sequence (after the second one) is the sum of the preceding two terms. Mathematically,

$$F_1 = F_2 = 1$$
$$F_{n+1} = F_n + F_{n-1} \ \forall \ n > 2$$

The Lucas sequence has the same recursive sequence relationship as the Fibonacci sequence. Mathematically,

$$L_1 = 1; L_2 = 3$$
$$L_{n+1} = L_n + L_{n-1} \ \forall \ n > 2$$

The first few Lucas numbers are

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ...$$

We defined the General Fibonacci sequence (G sequence) as the sequence which has the same recursive sequence relationship as the Fibonacci sequence, where each term in the sequence (after the second one) is the sum of the preceding two terms. Mathematically,

$$A_1 = x; A_2 = y$$
$$A_{n+1} = A_n + A_{n-1} \ \forall \ n > 2$$

The first few G numbers can be represented as

$$x, y, x + y, x + 2y, 2x + 3y, 3x + 5y, 5x + 8y, 8x + 13y, ...$$

2 Sum Identities

2.1 Sum Identities for Fibonacci Numbers

2.1.1 Sum of first n terms

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1$$

Proof. Base case $n = 1$: If $n = 1$, the left hand side is $F_1 = 1$ and the right hand side is $F_{1+2} - 1 = 2 - 1 = 1$. So, the theorem holds when $n = 1$.
Inductive hypothesis: Suppose the theorem holds for all values of n up to some k, $k \geq 1$.
Inductive step: Let $n = k + 1$. Then our left side is

$$\begin{aligned}\sum_{i=1}^{k+1} F_i &= F_{k+1} + \sum_{i=1}^{k} F_i \\ &= F_{k+1} + F_{k+2} - 1, \text{(by our inductive hypothesis)} \\ &= F_{k+3} - 1, (\because \ F_{k+1} + F_{k+2} = F_{k+3})\end{aligned}$$

which is our right side. So, the theorem holds for $n = k + 1$. By the principle of mathematical induction, the theorem holds for all $n \in \mathbb{N}$. □

2.1.2 Sum of first n even terms

Theorem 2.1.

$$\sum_{i=0}^{n} F_{2i} = F_{2n+1} - 1$$

Proof. Base case $n = 1$: If $n = 1$, the left hand side is $F_{2(1)} = F_2 = 1$ and the right hand side is $F_{2(1)+1} - 1 = F_3 - 1 = 2 - 1 = 1$. So, the theorem holds when $n = 1$.
Inductive hypothesis: Suppose the theorem holds for all values of n up to some k, $k \geq 1$.
Inductive step: Let $n = k + 1$. Then our left side is

$$\begin{aligned}\sum_{i=1}^{k+1} F_{2i} &= F_{2(k+1)} + \sum_{i=1}^{k} F_{2i} \\ &= F_{2k+2} + F_{2k+1} - 1, \text{(by our inductive hypothesis)} \\ &= F_{2k+3} - 1, (\because\ F_{2k+1} + F_{2k+2} = F_{2k+3}\ ) \\ &= F_{2(k+2)+1} - 1\end{aligned}$$

which is our right side. So, the theorem holds for $n = k + 1$. By the principle of mathematical induction, the theorem holds for all $n \in \mathbb{N}$. ☐

2.1.3 Sum of first n odd terms

Theorem 2.2.

$$\sum_{i=0}^{n} F_{2i+1} = F_{2n+2}$$

Proof. Base case $n = 0$: If $n = 0$, the left hand side is $F_{2(0)+1} = F_1 = 1$ and the right hand side is $F_{2(0)+2} = F_2 = 1$. So, the theorem holds when $n = 0$.
Inductive hypothesis: Suppose the theorem holds for all values of n up to some k, $k \geq 1$.
Inductive step: Let $n = k + 1$. Then our left side is

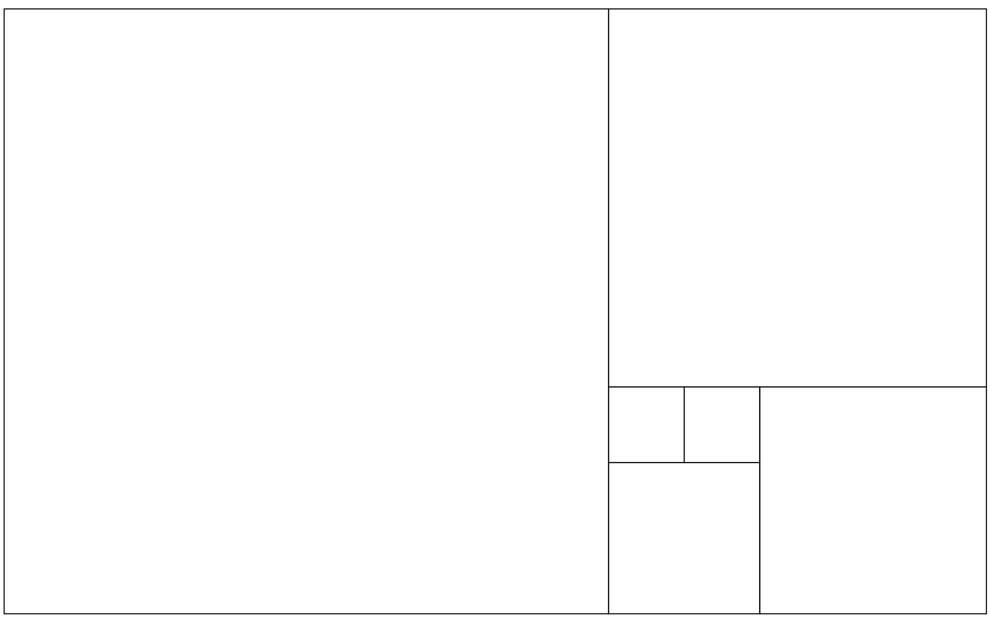
$$\begin{aligned}\sum_{i=0}^{k+1} F_{2i+1} &= F_{2(k+1)+1} + \sum_{i=0}^{k} F_{2i+1} \\ &= F_{2k+3} + F_{2k+2}, \text{(by our inductive hypothesis)} \\ &= F_{2k+4}, (\because\ F_{2k+3} + F_{2k+2} = F_{2k+4}\ ) \\ &= F_{2(k+2)}\end{aligned}$$

which is our right side. So, the theorem holds for $n = k + 1$. By the principle of mathematical induction, the theorem holds for all $n \in \mathbb{N}$. ☐

2.1.4 Sum of square of first n terms

Theorem 2.3.

$$\sum_{i=1}^{n} (F_i)^2 = F_n \times F_{n+1}$$



The dimensions of the rectangle formed by putting all of the squares together are F_n and $(F_n + F_{n-1}) = F_{n+1}$. So the area, i.e. the sum of all the squares till n is $F_n \times F_{n+1}$

Proof. Base case $n = 1$: If $n = 1$, the left hand side is $F_1^2 = 1$ and the right hand side is $F_1 \times F_{1+1} = F_1 \times F_2 = 1 \times 1 = 1$. So, the theorem holds when $n = 1$.
Inductive hypothesis: Assume the theorem holds true for all values of n up to some k, $k \geq 1$.
Inductive step: Let $n = k + 1$. Then our left side is

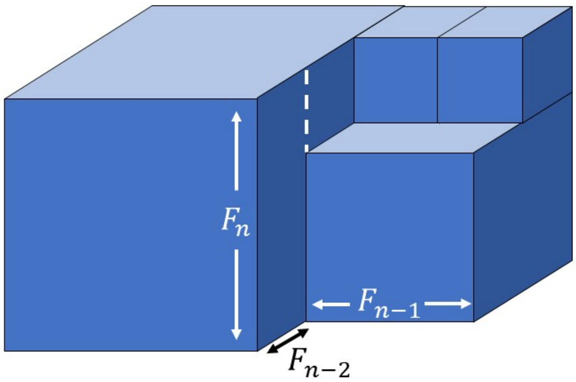
$$\begin{aligned}\sum_{i=1}^{k+1}(F_i)^2 &= \sum_{i=1}^{k}(F_i)^2 + F_{k+1}^2 \\ &= F_kF_{k+1} + F_{k+1}^2 \\ &= F_{k+1}(F_k + F_{k+1}) \\ &= F_{k+1}F_{k+2}\end{aligned}$$

which is our right hand side. So, the theorem holds true for $n = k + 1$. By the principle of mathematical induction, the theorem holds for all $n \in \mathbb{N}$. □

2.1.5 Sum of cubes of first n terms

Theorem 2.4.

$$S_n = \sum_{i=1}^{n} F_i^3 = \frac{3}{2}F_n^2F_{n+1} - F_n^3 - \frac{1}{2}F_{n-1}^3 + \frac{1}{2}$$



<u>*Pictorial Idea:*</u>
The volume of the above figure could be generalised as:

$$\sum_{i=1}^{n} F_i^3 = F_n^2F_{n+1} - (F_nF_{n-1}F_{n-2}) - (F_{n-1}F_{n-2}F_{n-3}) - ... - (F_3F_2F_1) \quad (1)$$

Now, we know that

$$\begin{aligned}F_n &= F_{n-1} + F_{n-2} \\ \implies F_{n-1} &= F_n - F_{n-2}\end{aligned}$$

Cubing both the sides,

$$\begin{aligned}(F_{n-1})^3 &= (F_n - F_{n-2})^3 \\ &= (F_n)^3 - 3F_nF_{n-2}(F_n - F_{n-2}) - (F_{n-2})^3\end{aligned}$$

$$= (F_n)^3 - 3F_nF_{n-1}F_{n-2} - (F_{n-2})^3$$
$$\Longrightarrow F_nF_{n-1}F_{n-2} = \frac{1}{3}(F_n^3 - F_{n-1}^3 - F_{n-2}^3)$$

Substituting the above expression in equation (1),

$$S_n = F_n^2F_{n+1} - [(F_nF_{n-1}F_{n-2}) + (F_{n-1}F_{n-2}F_{n-3}) + ... + (F_3F_2F_1)]$$
$$= F_n^2F_{n+1} - \frac{1}{3}[F_n^3 - F_{n-1}^3 - F_{n-2}^3 + F_{n-1}^3 - F_{n-2}^3 - F_{n-3}^3 + ... + F_3^3 - F_2^3 - F_1^3]$$
$$= F_n^2F_{n+1} - \frac{1}{3}[F_n^3 - (F_{n-2}^3 + F_{n-3}^3 + F_{n-3}^3 + ... + 2F_2^3 + F_1^3)]$$
$$\Longrightarrow 3S_n + F_n^3 + F_{n-1}^3 = 3F_n^2F_{n+1} - F_n^3 + S_n + 1$$
$$\Longrightarrow 2S_n = 3F_n^2F_{n+1} - 2F_n^3 - F_{n-1}^3 + 1$$
$$\Longrightarrow S_n = \frac{3}{2}F_n^2F_{n+1} - F_n^3 - \frac{1}{2}F_{n-1}^3 + \frac{1}{2}$$

Proof. Base case $n = 1$: on LHS we have $F_1^3 = 1$. On RHS we have $\frac{3}{2}F_1^2F_2 - F_1^3 - \frac{1}{2}F_0^3 + \frac{1}{2} = \frac{3}{2} - 1 - 0 + \frac{1}{2} = 1$. So, the theorem holds true for $n = 1$.

Assume it is true for $n = k$:

$$\sum_{i=1}^{k} F_i^3 = \frac{3}{2}F_k^2F_{k+1} - F_k^3 - \frac{1}{2}F_{k-1}^3 + \frac{1}{2}$$
$$\sum_{i=1}^{k+1} F_i^3 = \sum_{i=1}^{k} F_i^3 + F_{k+1}^3$$
$$= \frac{3}{2}F_k^2F_{k+1} - F_k^3 - \frac{1}{2}F_{k-1}^3 + \frac{1}{2} + F_{k+1}^3$$

Expanding F_{k-1}^3:

$$= \frac{3}{2}F_k^2F_{k+1} - F_k^3 + [-\frac{1}{2}F_{k+1}^3 + \frac{3}{2}F_{k+1}^2F_k - \frac{3}{2}F_k^2F_{k+1} + \frac{1}{2}F_k^3] + \frac{1}{2} + F_{k+1}^3$$
$$= -\frac{1}{2}F_k^3 + \frac{3}{2}F_{k+1}^2F_k + \frac{1}{2} + \frac{1}{2}F_{k+1}^3$$

Writing F_k as $F_{k+2} - F_{k+1}$ in the second term:

$$= -\frac{1}{2}F_k^3 + \frac{3}{2}F_{k+1}^2(F_{k+2} - F_{k+1}) + \frac{1}{2} + \frac{1}{2}F_{k+1}^3$$
$$= -\frac{1}{2}F_k^3 + \frac{3}{2}F_{k+1}^2F_{k+2} - \frac{3}{2}F_{k+1}^3 + \frac{1}{2} + \frac{1}{2}F_{k+1}^3$$
$$= -\frac{1}{2}F_k^3 + \frac{3}{2}F_{k+1}^2F_{k+2} - F_{k+1}^3 + \frac{1}{2}$$
$$= \frac{3}{2}F_{k+1}^2F_{k+2} - F_{k+1}^3 - \frac{1}{2}F_k^3 + \frac{1}{2}$$

Which is the RHS, so, the theorem holds true for $n = k + 1$. By the principle of mathematical induction, the theorem holds for all $n \in \mathbb{N}$. □

2.1.6 Sum of Sums

$\sum_{k=1}^{n}\sum_{i=1}^{k} F_i$ can be written in two different ways as

$$\sum_{k=1}^{n}(F_{k+2} - 1) = \sum_{i=1}^{n+2} F_i - 2 - n = F_{n+4} - 3 - n$$

and

$$\sum_{i=1}^{n}(n - i + 1)F_i = (n + 1)\sum_{i=1}^{n} F_i - \sum_{i=1}^{n} iF_i$$
$$= (n + 1)(F_{n+2} - 1) - \sum_{i=1}^{n} iF_i$$

Equating both gives

$$F_{n+4} - 3 - n = nF_{n+2} + F_{n+2} - n - 1 - \sum_{i=1}^{n} iF_i$$

$$\boxed{\sum_{i=1}^{n}(iF_i) = (n+1)F_{n+2} - F_{n+4} + 2}$$

2.2 Sum Identities for Lucas Numbers

2.2.1 Sum of first n terms

Theorem 2.5.

$$\sum_{i=1}^{n} L_i = L_{n+2} - 3$$

Proof. Base case $n = 1$: If $n = 1$, the left hand side is $L_1 = 1$ and the right hand side is $L_{1+2} - 3 = 4 - 3 = 1$. So, the theorem holds when $n = 1$.
Inductive hypothesis: Suppose the theorem holds for all values of n up to some k, $k \geq 1$.
Inductive step: Let $n = k + 1$. Then our left side is

$$\begin{aligned}\sum_{i=1}^{k+1} L_i &= L_{k+1} + \sum_{i=1}^{k} L_i \\ &= L_{k+1} + L_{k+2} - 3, \text{ (by our inductive hypothesis)} \\ &= L_{k+3} - 3, \ (\because\ L_{k+1} + L_{k+2} = L_{k+3}\ )\end{aligned}$$

which is our right side. So, the theorem holds for $n = k + 1$. By the principle of mathematical induction, the theorem holds for all $n \in \mathbb{N}$. □

2.2.2 Sum of first n even terms

Theorem 2.6.

$$\sum_{i=1}^{n} L_{2i} = L_{2n+1} - 1$$

Proof. Base case $n = 1$: If $n = 1$, the left hand side is $L_{2(1)} = L_2 = 3$ and the right hand side is $L_{2(1)+1} - 1 = L_3 - 1 = 4 - 1 = 3$. So, the theorem holds when $n = 1$.
Inductive hypothesis: Suppose the theorem holds for all values of n up to some k, $k \geq 1$.
Inductive step: Let $n = k + 1$. Then our left side is

$$\begin{aligned}\sum_{i=1}^{k+1} L_{2i} &= L_{2(k+1)} + \sum_{i=1}^{k} L_{2i} \\ &= L_{2k+2} + L_{2k+1} - 1, \text{ (by our inductive hypothesis)} \\ &= L_{2k+3} - 1, \ (\because\ L_{2k+1} + L_{2k+2} = L_{2k+3}\ ) \\ &= L_{2(k+2)+1} - 1\end{aligned}$$

which is our right side. So, the theorem holds for $n = k + 1$. By the principle of mathematical induction, the theorem holds for all $n \in \mathbb{N}$. □

2.2.3 Sum of first n odd terms

Theorem 2.7.

$$\sum_{i=0}^{n} L_{2i+1} = L_{2n+2} - 2$$

Proof. Base case $n = 0$: If $n = 0$, the left hand side is $L_{2(0)+1} = L_1 = 1$ and the right hand side is $L_{2(0)+2} - 2 = L_2 - 2 = 3 - 2 = 1$. So, the theorem holds when $n = 0$.
Inductive hypothesis: Suppose the theorem holds for all values of n up to some k, $k \geq 1$.
Inductive step: Let $n = k + 1$. Then our left side is

$$\sum_{i=0}^{k+1} L_{2i+1} = L_{2(k+1)+1} + \sum_{i=0}^{k} L_{2i+1}$$

$$
\begin{aligned}
&= L_{2k+3} + L_{2k+2} - 2, \text{ (by our inductive hypothesis)} \\
&= L_{2k+4} - 2, \ (\because \ L_{2k+3} + L_{2k+2} = L_{2k+4} \ ) \\
&= L_{2(k+2)} - 2
\end{aligned}
$$

which is our right side. So, the theorem holds for $n = k+1$. By the principle of mathematical induction, the theorem holds for all $n \in \mathbb{N}$. □

2.2.4 Sum of squares of first n terms

Theorem 2.8.

$$\sum_{i=0}^{n} (L_i)^2 = L_n \times L_{n+1} - 2$$

2.3 Sum Identities for G Numbers

2.3.1 Sum of first n terms

Theorem 2.9.

$$\sum_{i=1}^{n} A_i = A_{n+2} - y$$

Proof. Base case $n = 1$: If $n = 1$, the left hand side is $A_1 = x$ and the right hand side is $A_{1+2} - y = x + y - y = x$. So, the theorem holds when $n = 1$.
Inductive hypothesis: Suppose the theorem holds for all values of n up to some k, $k \geq 1$.
Inductive step: Let $n = k+1$. Then our left side is

$$
\begin{aligned}
\sum_{i=1}^{k+1} A_i &= A_{k+1} + \sum_{i=1}^{k} A_i \\
&= A_{k+1} + A_{k+2} - y, \text{ (by our inductive hypothesis)} \\
&= A_{k+3} - y, \ (\because \ A_{k+1} + A_{k+2} = A_{k+3} \ )
\end{aligned}
$$

which is our right side. So, the theorem holds for $n = k+1$. By the principle of mathematical induction, the theorem holds for all $n \in \mathbb{N}$. □

2.3.2 Sum of first n even terms

Theorem 2.10.

$$\sum_{i=0}^{n} A_{2i} = A_{2n+1} - x$$

Proof. Base case $n = 1$: If $n = 1$, the left hand side is $A_{2(1)} = A_2 = y$ and the right hand side is $A_{2(1)+1} - x = A_3 - x = x + y - x = y$. So, the theorem holds when $n = 1$.
Inductive hypothesis: Suppose the theorem holds for all values of n up to some k, $k \geq 1$.
Inductive step: Let $n = k+1$. Then our left side is

$$
\begin{aligned}
\sum_{i=1}^{k+1} A_{2i} &= A_{2(k+1)} + \sum_{i=1}^{k} A_{2i} \\
&= A_{2k+2} + A_{2k+1} - x, \text{ (by our inductive hypothesis)} \\
&= A_{2k+3} - x, \ (\because \ A_{2k+1} + A_{2k+2} = A_{2k+3} \ ) \\
&= A_{2(k+2)+1} - x
\end{aligned}
$$

which is our right side. So, the theorem holds for $n = k+1$. By the principle of mathematical induction, the theorem holds for all $n \in \mathbb{N}$. □

2.3.3 Sum of first n odd terms

Theorem 2.11.

$$\sum_{i=0}^{n} A_{2i+1} = A_{2n+2} + x - y$$

Proof. Base case $n = 0$: If $n = 0$, the left hand side is $A_{2(0)+1} = A_1 = x$ and the right hand side is $A_{2(0)+2} + x - y = A_2 + x - y = y + x - y = x$. So, the theorem holds when $n = 0$.
Inductive hypothesis: Suppose the theorem holds for all values of n up to some k, $k \geq 1$.
Inductive step: Let $n = k + 1$. Then our left side is

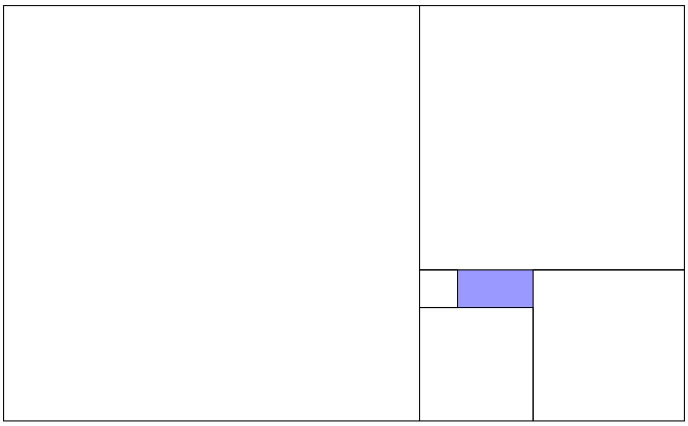
$$\begin{aligned}\sum_{i=0}^{k+1} A_{2i+1} &= A_{2(k+1)+1} + \sum_{i=0}^{k} A_{2i+1} \\ &= A_{2k+3} + A_{2k+2} + x - y, \text{ (by our inductive hypothesis)} \\ &= A_{2k+4} + x - y, \ (\because \ A_{2k+3} + A_{2k+2} = A_{2k+4}\ ) \\ &= A_{2(k+2)} + x - y\end{aligned}$$

which is our right side. So, the theorem holds for $n = k + 1$. By the principle of mathematical induction, the theorem holds for all $n \in \mathbb{N}$. □

2.3.4 Sum of squares of first n terms

$$\sum_{i=0}^{n} (A_i)^2 = A_n \times A_{n+1} - xy + x^2$$

The first two terms may not necessarily make a rectangle; there may be space left (shaded region below):



The dimensions of the entire rectangle will still be $A_n \times A_{n+1}$, but we will also have to subtract the area of the shaded region. The area of the shaded region will be:

$$\begin{aligned} & A_1 \times (A_2 - A_1) \\ &= x(y - x)\end{aligned}$$

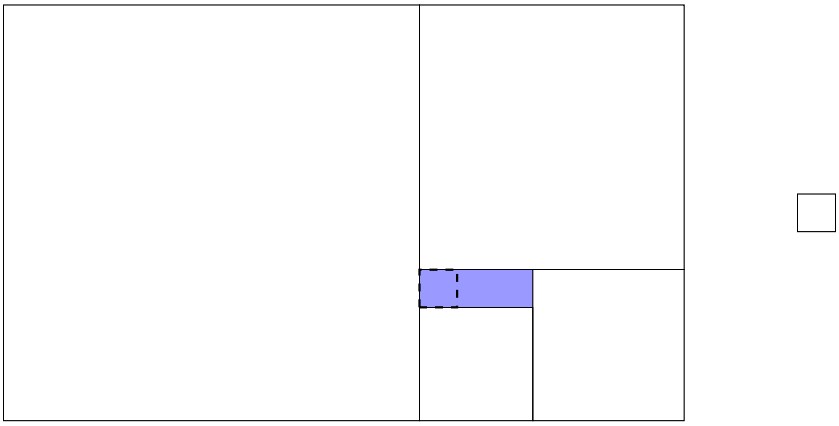
Now we have the area of the entire rectangle minus the area of the space left:

$$\sum_{i=1}^{n} A_i^2 = A_n \times A_{n+1} - x(y - x)$$

This can be rewritten as:

$$\sum_{i=1}^{n} A_i^2 = A_n \times A_{n+1} - xy + x^2$$

We can think of this as drawing the rectangle without the first term x, and then adding x^2 separately. This takes care of the case where $x > y$. $(-xy)$ is subtracting the area of the shaded region and $(+x^2)$ is adding back the first term which we had omitted in the drawing:



Proof. Base case $n = 1$: If $n = 1$, the left hand side is A_1^2 and the right hand side is $A_1A_{1+1} - A_1A_2 + A_1^2 = A_1(A_2 - A_2 + A_1) = A_1^2$. So, the theorem holds when $n = 1$.
Inductive hypothesis: Suppose the theorem holds true for all values of n up to some k, $k \geq 1$.
Inductive step: Let $n = k + 1$. Then our left side is

$$\begin{aligned}\sum_{i=1}^{k+1}(A_i)^2 &= \sum_{i=1}^{k}(A_i)^2 + A_{k+1}^2 \\ &= A_kA_{k+1} + A_{k+1}^2 + A_1A_2 + A_1^2 \\ &= A_{k+1}(A_k + A_{k+1}) + A_1A_2 + A_1^2 \\ &= A_{k+1}A_{k+2} + A_1A_2 + A_1^2\end{aligned}$$

which is our right hand side. So, the theorem holds true for $n = k + 1$. By the principle of mathematical induction, the theorem holds for all $n \in \mathbb{N}$. ☐

2.3.5 Relationship to Fibonacci numbers

$$A_n = (F_{n-2})x + (F_{n-1})y, n \geq 3$$

Proof. Base case $n = 3$: If $n = 3$, the left hand side is $A_3 = x + y$ and the right hand side is $(F_{3-2})x + (F_{3-1})y = (F_1)x + (F_2)y = x + y$.
So, the theorem holds when $n = 3$.
Inductive hypothesis: Suppose the theorem holds for all values of n up to some k, $k \geq 1$.
Inductive step: Let $n = k + 1$. Then our left side is

$$A_{k+1} = A_k + A_{k-1}$$

From our inductive hypothesis, we have

$$\begin{aligned}A_{k+1} &= (F_{k-2})x + (F_{k-1})y + (F_{k-3})x + (F_{k-2})y \\ &= (F_{k-2} + F_{k-3})x + (F_{k-1} + F_{k-2})y \\ &= (F_{k-1})x + (F_k)y \\ &= (F_{(k+1)-2})x + (F_(k + 1) - 1)y\end{aligned}$$

which is our right side. So, the theorem holds for $n = k + 1$. By the principle of mathematical induction, the theorem holds for all $n \geq 3$. ☐

3 General Fibonacci Sequence and the Golden Ratio

F_n	F_{n-1}	F_n/F_{n-1}
1	1	1
2	1	2
3	2	1.5
8	5	1.6
13	8	1.625
21	13	1.615384615
34	21	1.619047619
55	34	1.617647059
89	55	1.6181818182

We observe that as the Fibonacci numbers get larger, the ratio of F_n/F_{n-1} gets closer and closer to the Golden Ratio ($\phi \approx 1.6180$). Mathematically,

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \phi$$

Now, let's check if the same pattern follows for the Lucas sequence.

L_n	L_{n-1}	L_n/L_{n-1}
3	1	3
4	3	1.33334
7	4	1.75
11	7	1.57142858
18	11	1.63636363
29	18	1.61111111
47	29	1.620689656
76	47	1.617021277
123	76	1.618421053
199	123	1.617886179
322	199	1.618090452

We observe that as the Lucas numbers get larger, the ratio of L_n/L_{n-1} gets closer and closer to the Golden Ratio ($\phi \approx 1.6180$). Mathematically,

$$\lim_{n\to\infty} \frac{L_n}{L_{n-1}} = \phi$$

We can now generalize our observations in the form of the theorem stated below:

Theorem 3.1.

$$\lim_{n\to\infty} \frac{A_n}{A_{n-1}} = \phi \text{ , } \forall \; G \text{ - } sequences$$

Proof.

$$\begin{aligned}
\lim_{n\to\infty} \frac{A_{n+1}}{A_n} &= \lim_{n\to\infty} \frac{A_n}{A_{n-1}} = (say)\ x \ ...(1) \\
\lim_{n\to\infty} \frac{A_{n+1}}{A_n} &= \lim_{n\to\infty} \frac{A_n + A_{n-1}}{A_n} \\
&= \lim_{n\to\infty} (1 + \frac{A_n}{A_{n-1}}) \\
&= 1 + \frac{1}{\lim_{n\to\infty} \frac{A_{n-1}}{A_n}} \ ...(2)
\end{aligned}$$

From equation (1) and equation (2), we get

$$x = 1 + \frac{1}{x} \Longrightarrow x^2 - x - 1 = 0$$

Solving the quadratic equation for x, we get

$$x = \frac{1 \pm \sqrt{5}}{2}$$

As all A_n are positive,

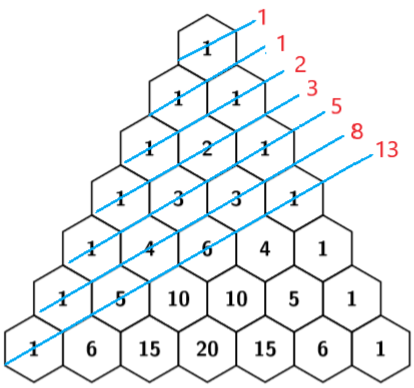
$$x = \frac{1 + \sqrt{5}}{2}$$

$$x = \lim_{n \to \infty} \frac{A_n}{A_{n-1}} = \phi$$

□

4 Fibonacci and Pascal's Triangle

Pascal's Triangle is formed by starting with an apex of 1. Every number below in the triangle is the sum of the two numbers diagonally above it to the left and the right, with positions outside the triangle counting as zero.
It is interesting to note that the numbers on diagonals of the triangle add to the Fibonacci sequence, as shown below.



4.1 Pascal's triangle in Binomial form

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

Fibonacci numbers as sums of diagonals:

$$F_n = \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i}{i-1}$$

The *ith* term in the sequence of terms that add to F_n:

$$F_{n_i} = \binom{n-i}{i-1} = \frac{(n-1)!}{(i-1)!(n-2i+1)!}$$

By the pattern in the Pascals triangle, F_{n_i} is formed by adding $F_{(n-1)_i}$ and $F_{(n-2)_{(i-1)}}$.

$$F_{(n-1)_i} + F_{(n-2)_{(i-1)}}$$

$$= \binom{n-i-1}{i-1} + \binom{n-i-1}{i-2}$$

$$= \frac{(n-i-1)!}{(i-1)!(n-2i)!} + \frac{(n-i-1)!}{(i-2)!(n-2i+1)!}$$

$$= \frac{(n-i-1)!(n-2i+1)}{(i-1)!(n-2i+1)!} + \frac{(n-i-1)!(i-1)}{(i-1)!(n-2i+1)!}$$

$$= \frac{(n-i-1)!(n-2i+1+i-1)}{(i-1)!(n-2i+1)!}$$

$$= \frac{(n-i-1)!(n-i)}{(i-1)!(n-2i+1)!}$$

$$= \frac{(n-i)!}{(i-1)!(n-2i+1)!}$$

$$= \binom{n-i}{i-1}$$

$$= F_{n_i}$$

Expanded forms of the summation of F_n, F_{n-1} and F_{n-2}:

$$F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + ... + \binom{n-\lfloor\frac{n+1}{2}\rfloor}{\lfloor\frac{n+1}{2}\rfloor - 1}$$

$$F_{n-1} = \binom{n-2}{0} + \binom{n-3}{1} + \binom{n-4}{2} + ... + \binom{n-\lfloor\frac{n}{2}\rfloor - 1}{\lfloor\frac{n}{2}\rfloor - 1}$$

$$F_{n-2} = \binom{n-3}{0} + \binom{n-4}{1} + \binom{n-5}{2} + ... + \binom{n-\lfloor\frac{n-1}{2}\rfloor - 2}{\lfloor\frac{n-1}{2}\rfloor - 1}$$

$F_{(n-1)_1} = F_{n_1}$, all subsequent F_{n_i} are formed by adding $F_{(n-1)_i}$ $F_{(n-2)_{i-1}}$

Case 1: n is Even

F_n and F_{n-1} will have the same number of terms. The last two terms of F_{n-2} and F_{n-1} will add to the last term of F_n.

Case 2: n is Odd

F_{n-1} and F_{n-2} will have the same number of terms. All terms add as before, but the last term of F_{n-2} is equal to the last term of F_n:

$F_{n_{(\lfloor\frac{n+1}{2}\rfloor}}$	$F_{(n-2)_{(\lfloor\frac{n+1}{2}\rfloor-1)}}$
$\binom{n-\lfloor\frac{n-1}{2}\rfloor - 2}{\lfloor\frac{n-1}{2}\rfloor - 1}$	$\binom{n-\lfloor\frac{n+1}{2}\rfloor}{\lfloor\frac{n+1}{2}\rfloor - 1}$

$$\binom{n-(\frac{n}{2}-\frac{1}{2})-2}{\frac{n}{2}-\frac{1}{2}-1} \qquad \binom{n-(\frac{n}{2}+\frac{1}{2})}{\frac{n}{2}+\frac{1}{2}-1}$$

$$\binom{\frac{n}{2}-1.5}{\frac{n}{2}-1.5} \qquad \binom{\frac{n}{2}-0.5}{\frac{n}{2}-0.5}$$

$$\Longrightarrow F_{n_{\lfloor \frac{n+1}{2} \rfloor}} = F_{(n-2)_{(\lfloor \frac{n+1}{2} \rfloor -1)}}$$

4.2 Pascal and Fibonacci recursion with starting x, y

We modify the Pascal's Triangle by starting with an apex of x. And the second row being y and x. Every number below in the triangle is the sum of the two numbers diagonally above it to the left and the right, with positions outside the triangle counting as zero just like the Pascal's Triangle.

x
y x
y x+y x
y x+2y 2x+y x
y x+3y 3x+3y 3x+y x
y x+4y 4x+6y 6x+4y 4x+y x
y x+5y 5x+10y 10x+10y 10x+5y 5x+y x

Observing the coefficients of x and y in this new triangle, we realize they are adjacent pairs of the Pascal's Triangle of the previous row.
Another way of looking at this is by splitting the x and the y terms, that is removing all the x's we just get the Pascal's Triangle times y.
It is interesting to note that from these modifications, the numbers on diagonals of this triangle sum up to the General Fibonacci numbers = G_n

5 Combinatorial Interpretation of Fibonacci Numbers

$n-1$	Compositions	No. of compositions	F_n
1	1	1	1
2	1+1 , 2	2	2
3	1+1+1 , 1+2 , 2+1	3	3
4	1+1+1+1 , 1+1+2 , 1+2+1 , 2+1+1 , 2+2	5	5
5	1+1+1+1+1, 2+1+1+1, 1+2+1+1, 1+1+2+1, 1+1+1+2, 2+2+1, 2+1+2, 1+2+2	8	8

From the above table, it is interesting to note that the number of compositions of $n-1$ with parts 1 or 2 is equal to the *nth* Fibonacci Number. Mathematically, this can be stated as the theorem below :

Theorem 5.1. *The Fibonacci number F_n is the number of compositions of $n-1$ into parts equal to 1 or 2.*

Proof. **1:** Let k be the number of twos in a (1,2)-composition of $n-1$
$\Longrightarrow$ The number of ones in this composition $= n-1-2k$

$$\text{Number of (1,2)-compositions of } n-1 \text{ for a given } k = \binom{\text{Total number of places}}{\text{Places in which the number is 2}}$$

$$= \binom{n-1-k}{k}$$

There can be a minimum of 0 twos (and all ones) in the composition to a maximum of all twos if $n-1$ is even, or a one and $\frac{n-2}{2}$ twos if $n-1$ is odd.

Summing the number of (1,2)-compositions for all k with limits $k=0$ to $\lfloor \frac{n-1}{2} \rfloor$ we get,

$$\text{Number of (1,2)-compositions of } n-1 = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k}$$

Taking $k = i-1$,

$$\text{Number of (1,2)-compositions of } n-1 = \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i}{i-1}$$

which from Section 4.1, Pascal's triangle in Binomial form $= F_n$
$\implies$ Number of (1,2)-compositions of $n-1 = F_n$ □

Proof. **2:**
Let the number of ways n-1 can be written as a sum of ones and twos be V_n $V_1 = 1$ $V_2 = 1$ ...(1)
For n>2, V_n = number of ways in which n-1 can be written as a sum of ones and twos such that the last (rightmost) number is 1
+ number of ways in which n-1 cam be written as sum of ones and twos such that the last number is 2
If the last number is 1, the sum of all previous numbers=n-2
$\implies$ number of ways to write n-1 as sum of ones and twos such that the last number is 1= number of ways to write the numbers before the last number $= V_{n-1}$
If the last number is 2, the sum of all previous numbers is n-3
$\implies$ number of ways to write n-1 as sum of ones and twos such that the last number is 2 = number of ways to write the numbers before the last number $=V_{n-2}$
Hence, $V_n = V_{n-1} + V_{n-2}$ ...(2)
So, from (1) and (2), $V_n = F_n$ □

6 Extension of Fibonacci recursion sequences into negative numbers

We know that Fibonacci recursion is:

$$F_{n+1} = F_n + F_{n-1} \ \forall \ n > 2$$

If we remove the constraint $n > 2$, we can extent G-sequences towards the left side (negative axis).

Fibonacci numbers, F_0 is boxed:

$$\ldots, 13, -8, 5, -3, 2, -1, 1, \boxed{0}, 1, 1, 2, 3, 5, 8, 13, \ldots$$

The numbers on the left hand side are the same as those on the right, but the sign of the numbers alternates.

Conjecture 6.1. $F_{-n} = (-1)^{n-1} F_n$

Lucas numbers, L_0 is boxed:

$$\ldots, -29, 18, -11, 7, -4, 3, -1, \boxed{2}, 1, 3, 4, 7, 11, 18, 29, \ldots$$

Conjecture 6.2. $L_{-n} = (-1)^n L_n$

Some terms of G sequence with starting x and y:

$$\ldots, \text{-3x+2y, 2x-y, } \boxed{-x+y}, \text{ x, y, x+y, } \ldots$$

Boxed term will be point of "symmetry"

(assuming the sign will always alternate)
(If there is symmetry for first few terms, does it follow that the entire sequence will be symmetric?)

There will be symmetry across F_0 when:
Case 1: Every other term is negative, starting with A_{-2}:

$$\begin{aligned} x &= 2x - y \text{ and } y = -(-3x + 2y) \\ \implies x &= y \end{aligned}$$

Case 2: Every other term is negative, starting with A_{-1}:

$$\begin{aligned} x &= -(2x - y) \text{ and } y = -3x + 2y \\ \implies y &= 3x \end{aligned}$$

Theorem 6.1. *For all G sequences where $x = y$: $A_{-n} = (-1)^{n-1} A_n$*

Proof.

$$\begin{aligned} A_1 &= A_2 \\ A_0 &= A_2 - A_1 \\ \implies A_0 &= 0 \\ A_1 &= A_0 + A_{-1} \\ \implies A_{-1} &= A_1 - A_0. \\ \implies A_{-1} &= A_1 \end{aligned} \tag{1}$$

Base case 1: $A_{-1} = (-1)^0 A_1 \implies A_{-1} = A_1$, which is true by (1)

$$\begin{aligned} \text{By definition } A_{-2} &= A_0 - A_{-1} \\ \implies A_{-2} &= -A_{-1} \\ \implies A_{-2} &= -A_1 \\ \implies A_{-2} &= -A_2 \end{aligned} \tag{2}$$

Base case 2: $A_{-2} = (-1)^1 A_2 \implies A_{-2} = -A_2$, which is true by (2)
Induction step : Assume it is true for $n = k$ and $n = k - 1$

$$\begin{aligned} A_{-(k+1)} + A_{-k} &= A_{-(k-1)} \\ A_{-(k+1)} &= A_{-(k-1)} - A_{-k} \\ &= (-1)^{k-2} A_{k-1} - (-1)^{k-1} A_k \\ &= (-1)^k (A_{k-1} + A_k) \\ &= (-1)^k A_{k+1} \end{aligned}$$

□

Theorem 6.2. *For all G sequences where $y = 3x$: $A_{-n} = (-1)^n A_n$*

Proof.

$$\begin{aligned} A_2 &= 3A_1 \\ A_0 + A_1 &= A_2 \\ \implies A_0 &= 2A_1 \\ A_{-1} + A_0 &= A_1 \\ \implies A_{-1} &= -A_1 \end{aligned} \tag{3}$$

Base case 1: $A_{-1} = (-1)^1 A_1 \implies A_{-1} = -A_1$, which is true by (3)

$$A_{-2} = A_0 - A_{-1}$$

$$\Longrightarrow A_{-2} = 2A_1 - (-A_1)$$
$$\Longrightarrow A_{-2} = 3A_1$$
$$\Longrightarrow A_{-2} = A_2 \tag{4}$$

Base case 2: $A_{-2} = (-1)^2 A_2 \Longrightarrow A_{-1} = A_2$, which is true by (4)
Induction step: Assume it is true for $n = k$ and $n = k - 1$

$$\begin{aligned} A_{-(k+1)} + A_{-k} &= A_{-(k-1)} \\ A_{-(k+1)} &= A_{-(k-1)} - A_{-k} \\ &= (-1)^{k-1} A_{k-1} - (-1)^k A_k \\ &= (-1)^{k-1}(A_{k-1} + A_k) \\ &= (-1)^{k+1} A_{k+1} \end{aligned}$$

□

7 Other Properties of Fibonacci Sequence

Consider the following table:

F_n	F_{n+1}	$\gcd(F_n, F_{n+1})$
1	1	1
1	2	1
2	3	1
3	5	1
5	8	1
8	13	1
13	21	1

We observe that greatest common divisor of any two consecutive numbers in the Fibonacci sequence is 1. Mathematically,

Lemma 7.1. $\gcd(F_n, F_{n+1}) = 1$

Proof. Base Case $n = 1$: If $n = 1$, the left hand side is $\gcd(F_1, F_2) = \gcd(1, 1) = 1$ which is equal to the right hand side. So, the lemma holds when $n = 1$.

Inductive hypothesis: Suppose the lemma holds for all values of n up to some k, $k \geq 1$.

Inductive step: Let $n = k + 1$.
Then our left side is

$$\begin{aligned} \gcd(F_{k+1}, F_{k+2}) &= \gcd(F_{k+1}, F_{k+2} - F_{k+1}) \text{ from Euclid's lemma} \\ &= \gcd(F_{k+1}, F_k) \\ &= 1 \text{ from induction hypothesis} \end{aligned}$$

which is our right side. So, the lemma holds for $n = k + 1$. By the principle of mathematical induction, the lemma holds for all $n \in \mathbb{N}$. □

Corollary 1. *F_n and F_{n+2} are coprime*

Proof. Let d be the gcd of (F_n, F_{n+2}).
We know that
$F_{n+2} = F_n + F_{n+1}$
$d|F_{n+2} \Longrightarrow d \mid F_n + F_{n+1}$
But $d|F_n$ because d is a factor of F_n
$\Longrightarrow d|F_{n+1}$
d is a factor of both F_n and F_{n+1}
d=1 from theorem 7.1 □

Theorem 7.2. $F_n = F_k \times F_{n-k+1} + F_{k-1} \times F_{n-k}$ *, where* $k \in [2, n-1]$

Proof. Base Case $k = 2$: If $k = 2$, the right hand side is $F_2 \times F_{n-1} + F_1 \times F_{n-2} = 1 \times F_{n-1} + 1 \times F_{n-2} = F_{n-1} + F_{n-2} = F_n$ which is equal to the left hand side. So, the theorem holds when $k = 2$.

Inductive hypothesis: Suppose the theorem holds for all values of k up to some m, $m \in [2, n-1]$.

Inductive step: Let $k = m + 1$. Then our right side is

$$\begin{aligned} F_m \times F_{n-m-1} + F_{n-m} \times F_{m+1} &= F_m(F_{n-m} + F_{n-m-1}) + F_{n-m}(F_{m+1} - F_m) \\ &= F_m \times F_{n-m+1} + F_{m-1} \times F_{n-m} \\ &= F_n \end{aligned}$$

which is our left side. So, the theorem holds for $k = m+1$. By the principle of mathematical induction, the theorem holds for all $k \in [2, n-1]$

□

Corollary 2. $F_{a+b} = F_a \times F_{b-1} + F_{a+1} \times F_b$

We get this corollary by taking n=a+b and k=a

Corollary 3. $F_{2n} = F_n(F_{n-1} + F_{n+1})$

We get this by taking a=n and b=n in the equation in corollary 2

Corollary 4. $F_{2n+1} = {F_n}^2 + (F_{n+1})^2$

We get this by taking a=n and b=n+1 in corollary 2

Theorem 7.3. $F_{n+mk} \equiv (F_{k-1})^m \times F_n \mod (F_k)$

Proof. Base Case $m = 1$: If $m = 1$, we know that $F_{n+k} \equiv F_{k-1} \times F_m \mod (F_k)$ from theorem 7.2. (Putting a=n and b=k in corollary 2) So, the theorem holds when $m = 1$.

Inductive hypothesis: Suppose the theorem holds for all values of m up to some a that is when $m \leq a$.

Inductive step: Let $m = a + 1$. Then our left side is

$$\begin{aligned} F_{n+ak+k} &\equiv F_{k-1} \times (F_{n+ak}) from putting a = n + ak and b = k in corollary 2 to theorem 7.2 \\ &\equiv F_{k-1} \times ((F_{k-1})^a \times F_n) \\ &= (F_{k-1})^{a+1} \times F_n \mod (F_k). \end{aligned}$$

which is our right side. So, the theorem holds for $m = a+1$. By the principle of mathematical induction, the theorem holds for all $m \in \mathbb{N}$.

□

Theorem 7.4. $F_m | F_n \Longleftrightarrow m | n$

Proof. Let $m | n \Longrightarrow n = km, k \in \mathbb{N}$ From theorem 7.3,

$$\begin{aligned} F_{m+(k-1)m} &\equiv {F_{k-2}}^m \times F_m \mod (F_m) \\ &\Longrightarrow F_m \mid F_n \end{aligned}$$

We have proved the if part. Now,
Suppose $m \leq n$ and let $m \nmid n$.

$$\Longrightarrow n = km + r \text{ ,such that } k, r > 0 \text{ and } r < m$$
$$\Longrightarrow F_n \equiv {F_{m-1}}^k \times F_r \mod F_m$$
$$\Longrightarrow F_m \mid F_n \Longleftrightarrow F_m \mid (F_{m-1})^k \times F_r$$

From lemma 6.1, $\gcd(F_{m-1}, F_m) = 1$.

$$\Longrightarrow 0 < F_r < F_m$$
$$\Longrightarrow F_m \nmid {F_{m-1}}^k \times F_r$$
$$\Longrightarrow F_m \nmid F_n$$

Therefore, we have proven the only if by contraposition. □

Theorem 7.5. $\gcd(F_m, F_n) = F_{\gcd(m,n)}$

Proof. We prove this by induction on m+n
Base Case: When m+n=2
$\gcd(F_1, F_1) = F_{\gcd(1,1)} = F_1$

Strong Inductive hypothesis: Suppose the theorem holds $\forall$ sums $< m + n$
Induction step: Wlog, $n > m$

$\gcd(F_m, F_n) = \gcd(F_m, F_{(n-m)+m})$
From Corollary 2 in Theorem 7.2,
$F_{(n-m)+m} = F_{n-m} \times F_{m-1} + F_{n-m+1} \times F_m$

$\gcd(F_m, F_n) = \gcd(F_m, F_{n-m} \times F_{m-1} + F_{n-m+1} \times F_m)$
$= \gcd(F_m, F_{n-m} \times F_{m-1})$ [By Euclid's Lemma]
$= \gcd(F_m, F_{n-m})$ [This is because $\gcd(F_m, F_{m-1}) = 1$ from lemma 7.1]
Now observe that $m + (n - m) = n < m + n$
So we can use our induction hypothesis to get
$\gcd(F_m, F_{n-m}) = F_{\gcd(m,n-m)}$
We know $\gcd(m, n - m) = \gcd(m, n)$
Hence, $\gcd(F_m, F_{n-m}) = \gcd(F_m, F_n) = F_{\gcd(m,n)}$
By the Principle of Mathematical Induction, the theorem holds $\forall$ sum of $m + n$ and hence $\forall m, n$

□

Theorem 7.6. ${F_{n-1}F_{n+1} - F_n}^2 = (-1)^n$

Proof. Base Case $n = 2$: If $n = 2$, the left hand side is $F_{2-1}F_{2+1} - {F_2}^2 = F_1F_3 - {F_2}^2 = 1(2) - 1 = 2 - 1 = 1$ and the right hand side is $(-1)^2 = 1$ So, the theorem holds when $n = 2$.

Inductive hypothesis: Suppose the theorem holds for all values of n up to some k, $k \geq 2$.

Inductive step: Let $n = k + 1$.

$$\begin{aligned} {F_k}^2 - F_{k+1}F_{k-1} &= {F_k}^2 - (F_{k+1} - F_k)F_{k+1} \\ &= {F_k}^2 + F_kF_{k+1} - {F_{k+1}}^2 \\ &= F_k(F_k + F_{k+1}) - {F_{k+1}}^2 \\ &= F_kF_{k+2} - {F_{k+1}}^2 \end{aligned}$$

From the inductive hypothesis, $F_{k-1}F_{k+1} - {F_k}^2 = (-1)^k$

$$\begin{aligned} &\Longrightarrow {F_k}^2 - F_{k-1}F_{k+1} = (-1)^{k+1} \\ &\Longrightarrow F_kF_{k+2} - {F_{k+1}}^2 = (-1)^{k+1} \end{aligned}$$

which is our right side. So, the theorem holds for $n = k + 1$. By the principle of mathematical induction, the theorem holds for all $n \geq 2$. □

Corollary 5. ${F_{n-1}}^2 + F_nF_{n-1} - {F_n}^2 = (-1)^n$

Proof. We prove for 3 ≤ n
We know $F_{k+1}F_{k-1} - F_k^2 = (-1)^k$: proved in theorem 7.5

For $k = n - 1$:

$$\begin{aligned} (F_{n-2})F_n - F_{n-1}^2 &= (-1)^{n-1} \\ \Longrightarrow F_n(F_n - F_{n-1}) - F_{n-1}^2 &= (-1)^{n-1} \\ \Longrightarrow F_n^2 - F_nF_{n-1} - F_{n-1}^2 &= (-1)^{n-1} \\ \Longrightarrow -F_n^2 + F_nF_{n-1} + F_{n-1}^2 &= (-1)^n \\ \Longrightarrow F_{n-1}^2 + F_nF_{n-1} - F_n^2 &= (-1)^n \end{aligned}$$

It is easy to check that the identity also works for n=2 □

Theorem 7.7. ${F_n}^2 - F_{n-k}F_{n+k} = {F_k}^2(-1)^{n-k}$

Proof. Please do not read this in a monochromatic PDF reader as this is a colored proof.

$$
\begin{aligned}
F_n^2 - F_{n-k}F_{n+k} &= (F_{n-k}F_{k+1} + F_{n-k-1}F_k)^2 - F_{n-k}(F_{n-k}F_{2k+1} + F_{n-k-1}F_{2k}) \\
&= (F_{n-k}F_{k+1} + F_{n-k-1}F_k)^2 - F_{n-k}(F_{n-k}F_{2k+1} + F_{n-k-1}F_{2k}) \\
&= (F_{n-k}F_{k+1} + F_{n-k-1}F_k)^2 - F_{n-k}(F_{n-k}[F_k^2 + F_{k+1}^2] + F_{n-k-1}[F_kF_{k-1} + F_kF_{k+1}]) \\
&= F_{n-k}^2F_{k+1}^2 + F_{n-k-1}^2F_k^2 + 2F_{n-k}F_{k+1}F_{n-k-1}F_k - F_{n-k}^2F_k^2 - F_{n-k}^2F_{k+1}^2 \\
&\quad - F_{n-k}F_{n-k-1}F_kF_{k-1} - F_{n-k}F_{n-k-1}F_kF_{k+1} \\
&= F_{n-k}^2F_{k+1}^2 + F_{n-k-1}^2F_k^2 + 2F_{n-k}F_{k+1}F_{n-k-1}F_k - F_{n-k}^2F_k^2 - F_{n-k}^2F_{k+1}^2 \\
&\quad - F_{n-k}F_{n-k-1}F_kF_{k-1} - F_{n-k}F_{n-k-1}F_kF_{k+1} \\
&= F_{n-k-1}^2F_k^2 + (F_{n-k}F_{n-k-1}F_kF_{k+1} - F_{n-k}F_{n-k-1}F_kF_{k-1}) - F_{n-k}^2F_k^2 \\
&= F_{n-k-1}^2F_k^2 + F_{n-k}F_{n-k-1}F_k(F_{k+1} - F_{k-1}) - F_{n-k}^2F_k^2 \\
&= F_{n-k-1}^2F_k^2 + F_{n-k}F_{n-k-1}F_k^2 - F_{n-k}^2F_k^2 \\
&= F_k^2(F_{n-k-1}^2 - F_{n-k}^2 + F_{n-k}F_{n-k-1}) \\
&= F_k^2(-1)^{n+k}
\end{aligned}
$$

We get this from corollary 2 by putting a=k and b=n-k
We get this from corollary 2 by putting a=2k and b=n-2k
We get this from corollary 4
We get this from corollary 5
We get this from expanding
We get this from expanding
We get this from corollary 5 □

8 Fibonacci Sequence Modulus m

Under this section, we talk about the some properties of the Fibonacci Sequence under a modulus. Let us first define some terms that will be frequently used in this section.

Definition 8.1. $F \pmod{m}$ means the sequence of the least non-negative residues of the terms of the Fibonacci sequence taken modulo m.

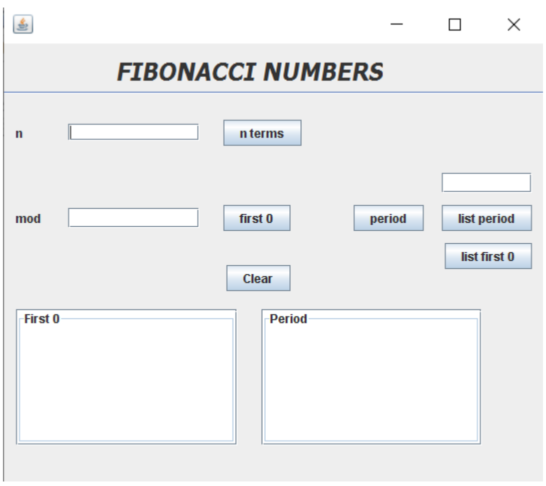
For example: $F \pmod 3 = 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, \ldots$

Definition 8.2. Let $Z(m)$ denote the position of the first zero in the sequence $F \pmod{m}$. Then, $Z(m)$ is the least positive integer k such that $F_k \equiv 0$.

For example, $Z(3) = 4$

Definition 8.3. Let $\tau(m)$ denote the period of $F \pmod{m}$. Then, $\tau(m)$ is the least positive integer k such that $F_k \equiv 0$ and $F_{k+1} \equiv 1 \pmod{m}$.

For example,$\tau(m)8$ We developed the following JAVA program to compute $\tau(m)$ and $z(m)$. *Refer to Appendix 1 for the source code.*



With the help of the program, we calculated $\tau(m)$ and $Z(m)$ which are listed below in the form of the table for $2 \leq m \leq 15$.

m	Period of F (mod m), $\tau(m)$	First zero in F (mod m), $Z(m)$
2	3	3
3	8	4
4	6	6
5	20	5
6	24	12
7	16	8
8	12	6
9	24	12
10	60	15
11	10	10
12	24	12
13	28	7
14	48	24
15	40	20

From the above table we observe many interesting results which are stated as the theorems below. We also find few interesting patterns that can be stated as conjectures.

Observation 8.1. $F(mod$ m) is periodic.

It will be intriguing to note that this lemma can be generalised for the G-sequence as well.

Theorem 8.1. *$G(mod$ m) is periodic.*

Proof. In mod m, there are m different numbers possible in the $G(mod$ m), so the number of different x and y starting pairs we can have is m^2.

Assume that $G(mod$ m) is not periodic. This means that any pair that appears in $G(mod$ m) will appear only once which will lead to infinitely many unique pairs of terms. But there are only m^2 different pairs possible. ($\Rightarrow \Leftarrow$)

$\Longrightarrow$ At least one pair will be repeated.
Assume some a,b repeat, that is, $A_k \equiv a, A_{k+1} \equiv b$ and $A_n \equiv a, A_{n+1} \equiv b$ for some k<m^2 and k<n
We want to prove $A_{n-k+1} \equiv x A_{n-k+2} \equiv y$ by induction.
We prove $A_{n-i+1} \equiv A_{k-i+1}$ by induction.
Base case: It is true for $i = 1$ and $i = 2$
Inductive hypothesis: Assume it is true for i=m and i=m+1
Inductive step: $A_{n+1-[m+2]} = A_{n+1-m} - A_{n+1-m-1} \equiv A_{k+1-m} - A_{k+1-m-1} = A_{k-i-m+1}$ ☐

Theorem 8.2. *There will never be a period of 2 mod n*

Proof. Say we start the recursion with any two numbers x and y:

$$x, \quad y, \quad x+y, \quad x+2y, \quad ....$$

To prove that this recursion (mod n) will never repeat with a period 2, we can show that if term(1) and term(3) are congruent to each other mod n then term(2) and term(4) will not be congruent to each other mod n. ☐

Lemma 8.3. *if $x \equiv x+y \mod n$ then $y \not\equiv x+2y \mod n$, where x and y are not both equal to 0.*

Proof.

$$\begin{aligned} & x \equiv x+y \mod n \\ \Longrightarrow & y \equiv 0 \mod n \\ assume: & \\ & y \equiv x+2y \mod n \\ \Longrightarrow & x \equiv 0 \mod n \end{aligned}$$

But x and y cannot *both* be equal to 0 ($\Rightarrow\Leftarrow$) □

8.1 Fibonacci sequence mod n

Lemma 8.4. *$m = \tau(n)k + r \Longrightarrow F_m \equiv F_r \mod n$ if $r \neq 0$ and $F_m \equiv 0 \mod n$ if $r = 0$*

Theorem 8.5. *m,n are coprime $\Longrightarrow \tau(mn) =$ LCM of $\tau(m)$ and $\tau(n)$*

Proof.

$$\begin{aligned} F_{\tau(mn)} &\equiv 0 \text{ mod mn and } F_{\tau(mn)+1} \equiv 1 \text{ mod mn} \\ \Longrightarrow F_{\tau(mn)} &\equiv 0 \text{ mod mn and } F_{\tau(mn)+1} \equiv 1 \text{ mod mn} \end{aligned}$$

Also, $F_{\tau(mn)} \equiv 0$ mod n $F_{\tau(mn)+1} \equiv 1$ mod n
$\Longrightarrow \tau(m)|\tau(mn)$ and $\tau(n)|\tau(mn)$

$\because$ of the way $\tau(mn)$ is defined, it is the least positive integer such that it is divisible by both $\tau(m)$ and $\tau(n)$ □

Lemma 8.6. *For some fixed value of k, $F_a \equiv {F_{Z(n)-1}}^k F_r \mod n$ is true for $r = 1$ and $r = 2 \Longrightarrow$ it is true $\forall 0 < r < Z(n)$*

Proof. We prove this by induction.
Base case: It is true for r=1 and r=2
Induction step: Assume it is true for r=m and r=m-1

$$\begin{aligned} F_{Z(n)k+m-1} &\equiv F^k_{Z(n)-1} F_{m-1} \mod n \\ F_{Z(n)k+m} &\equiv F^k_{Z(n)-1} F_m \mod n \\ \Longrightarrow F_{Z(n)k+m-1} + F_{Z(n)k+m} &\equiv F^k_{Z(n)-1} F_{m-1} + F_{Z(n)-1} F_m = F^k_{Z(n)-1}[F_{m-1} + F_m] \mod n \\ \Longrightarrow F_{Z(n)k+m+1} &\equiv F^k_{Z(n)-1} + F_{m+1} \mod n \end{aligned}$$

□

Theorem 8.7. *If $a = k \times Z(n) + r$ where $r < Z(n)$ and r is non-negative. Then, $F_a \equiv {F_{Z(n)-1}}^k F_r$* mod n

Proof. We prove this by induction.
Base case: It is true for k=1 because $F_{Z(n)+1} \equiv F_{Z(n)-1} F_1 \mod n$ and $F_{Z(n)+2} \equiv F_{Z(n)-1} F_2$ mod n
Inductive hypothesis: Assume it is true for $k = m$
Induction step:

$$\begin{aligned} F_{Z(n)[m+1]} &= F_{Z(n)m+Z(n)-2} + F_{Z(n)m+Z(n)-1} \\ \Longrightarrow F_{Z(n)[m+1]} &\equiv F^m_{Z(n)-1}[F_{Z(n)-2} + F_{Z(n)-1}] \\ &\equiv F^m_{Z(n)-1}[F_{Z(n)}] \mod n \\ \Longrightarrow F_{Z(n)[m+1]+1} &= F_{Z(n)[m+1]} + F_{Z(n)m+Z(n)-1} \\ &\equiv F^m_{Z(n)-1}[F_{Z(n)}] + F^m_{Z(n)-1}[F_{Z(n)}] \\ &= F^m_{Z(n)-1} F_{Z(n)+1} \\ &\equiv {F_{Z(n)-1}}^{m+1} F_1 \end{aligned}$$

$$
\begin{aligned}
\implies F_{Z(n)[m+1]+2} &= F_{Z(n)[m+1]} + F_{Z(n)[m+1]+1} \\
&\equiv F^{m}_{Z(n)-1} F_{Z(n)} + F^{m+1}_{Z(n)-1} F_1 \\
&= F^{m}_{Z(n)-1} [F_{Z(n)} + F_{Z(n)-1} F_1] \\
&\equiv F^{m+1}_{Z(n)-1} F_2 \mod n
\end{aligned}
$$

□

Corollary 6. $m|F_n \iff Z(m)|n$

Theorem 8.8. $n|F_k + (-1)^k \times F_{\tau(n)-k}$

Proof. We prove this by induction
Base case: $k = 1$
We need to prove $F_1 \equiv F_{\tau(n)-1}$ mod n
This is true because $F_{\tau(n)-1} = F_{\tau(n)+} - F_{\tau(n)}$
$k = 2$ $F_{\tau(n)} - F_{\tau(n)-1} \equiv 1$ mod n $\implies F_{\tau(n)-2} \equiv F_2$ mod n

Inductive hypothesis:
Assume the statement is TRUE for $k = m - 1$ and $k = m$
$F_{m+1} = F_{m-1} + F_m \equiv -1[(-1)^{m-1} F_{\tau(n)-m+1} + (-1)^m F_{\tau(n)-m} = -1^m [F_{\tau(n)-m+1} - F_{\tau(n)-m}] = -1^m [F_{\tau(n)-m-1}]$
$\implies$ the statement is true for $k = m + 1$ □

Theorem 8.9. $\frac{\tau(m)}{Z(m)} = k$ *where* k *is the least positive number that satisfies* ${F_{Z(m)-1}}^k \equiv 1 \mod n$

Proof.

$m \mid F_{\tau(m)} \implies Z(m) \mid \tau(m)$
Let $\tau(m) = k \times Z(n)$
$F_{\tau(m)+1} = F_{k \times Z(m)+1} \equiv 1 \mod m$
From theorem 8.9 , $F_{\tau(m)+1} \equiv {F_{Z(m)-1}}^k \times F_1 \mod m$
${F_{Z(m)_1}}^k \equiv 1$ is the smallest number such that $F_{\tau(m)} \equiv 0 \mod m$ and $F_{\tau(m)+1}$, k has to be the smallest natural number such that the previous statement holds true

□

Theorem 8.10. $\frac{\tau(m)}{Z(m)} \mid 4$

Proof.

$$
\begin{aligned}
F_{Z(m)-1} &= F_{Z(m)+1} - F_{Z(m)} \\
\implies F_{Z(m)-1} &\equiv F_{Z(m)+1} \mod m
\end{aligned}
$$

From theorem 7.5,

$$
\begin{aligned}
F_{Z(m)-1} F_{Z(m)+1} - F_{Z(m)} &= (-1)^{Z(m)} \\
F^2_{Z(m)-1} &\equiv (-1)^{Z(m)} \\
\implies F^4_{Z(m)-1} &\equiv 1 \mod m
\end{aligned}
$$

□

<u>Note</u> : Z(m) is even $\implies \frac{\tau(m)}{Z(m)}$ is 1 or 2 and Z(m) is odd $\implies \frac{\tau(m)}{Z(m)}$ is 4

Theorem 8.11. $\forall m \geq 3$, $\tau(m)$ *is even*

Proof. Assume $\tau(m)$ is even and $\tau(m) \geq 6$.
From the theorem 8.9,

$$
\begin{aligned}
\frac{\tau(m)}{2} &= \tau(m) - \frac{\tau(m)}{2} \text{ cannot be even} \\
\implies \frac{\tau(m)}{2} &\text{ is odd.}
\end{aligned}
$$

$$\implies \frac{\tau(m)}{2} - 1 \text{ is even and } \frac{\tau(m)}{2} - 2 \text{ is odd.}$$

$$\implies F_{\frac{\tau(m)}{2}-1} \equiv -F_{\frac{\tau(m)}{2}+1} (mod\ p) \text{ and } F_{\frac{\tau(m)}{2}-2} \equiv F_{\frac{\tau(m)}{2}+2} (mod\ m)$$

$$\implies F_{\frac{\tau(m)}{2}-1} + F_{\frac{\tau(m)}{2}-2} \equiv -F_{\frac{\tau(m)}{2}+1} + F_{\frac{\tau(m)}{2}+2} (mod\ m)$$

$$\text{But } F_{\frac{\tau(m)}{2}-1} + F_{\frac{\tau(m)}{2}-2} = F_{\frac{\tau(m)}{2}} = F_{\frac{\tau(m)}{2}+1} + F_{\frac{\tau(m)}{2}+2}$$

$$\implies F_{\frac{\tau(m)}{2}} = F_{\frac{\tau(m)}{2}+1} + F_{\frac{\tau(m)}{2}+2} \equiv -F_{\frac{\tau(m)}{2}+1} + F_{\frac{\tau(m)}{2}+2} (mod\ m)$$

$$\implies m | 2 \times F_{\frac{\tau(m)}{2}+1} \qquad (\Rightarrow\Leftarrow)$$

$\tau(m)$ cannot be 2 because no numbers greater than 2 divide F_2.
$\tau(m)$ cannot be 4 because the only number greater than 2 that divides F_4 is 3 and $\tau(3) \neq 4$. □

Conjecture 8.1. If p is a prime, then $\tau(p^n) = \tau(p) \times p^{n-1}$

9 Linear Transformations

We get a new type of sequence by further generalizing the recursion as follows,

$$X_n = aX_{n-2} + bX_{n-1}$$

That is, each term in the sequence (after the second one) is the weighted sum of the preceding two terms. With weights a and b and the starting terms as x and y.
The first few terms of this sequence can be represented as

$$x, y, ax + by, abx + (a + b^2)y, (a^2 + ab^2)x + (2ab + b^3)y, ...$$

9.1 Matrix representation of Fibonacci Numbers

Looking at the Fibonacci sequence, we realize that a column matrix of a pair of consecutive Fibonacci Numbers can be represented as the linear transformation of the previous pair of the column matrix. Mathematically,

Lemma 9.1.

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} \qquad \dots (1)$$

Proof. The right hand side is

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} = \begin{bmatrix} 1 * F_{n-1} + 1 * F_{n-2} \\ 1 * F_{n-1} + 0 * F_{n-2} \end{bmatrix}$$
$$= \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$$

which is equal to our left hand side. □

Theorem 9.2. $\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}$

Proof. Base Case $n = 2$: If $n = 2$, the left hand side is $\begin{bmatrix} F_2 \\ F_1 \end{bmatrix}$ and the right hand side is

$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{0} \begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}$ So, the theorem holds when $n = 2$.

Inductive hypothesis: Suppose the theorem holds for some $n = k$, $k \geq 2$.

Inductive step: Let $n = k + 1$.
Then our left hand side is

$$\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k-2} \begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k-1} \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}$$

which is our right side. So, the theorem holds for $n = k + 1$. By the principle of mathematical induction, the theorem holds for all $n \in \mathbb{N}$. □

We observe that we can generalize the above theorem for the any X_n.

9.2 Matrix representation of a General Recursion

By extending Lemma 1 for $X_n = aX_{n-2} + bX_{n-1}$, we get,

Lemma 9.3.

$$\begin{bmatrix} X_n \\ X_{n-1} \end{bmatrix} = \begin{bmatrix} b & a \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X_{n-1} \\ X_{n-2} \end{bmatrix} \qquad \ldots (2)$$

Proof. The right hand side is

$$\begin{bmatrix} b & a \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X_{n-1} \\ X_{n-2} \end{bmatrix} = \begin{bmatrix} b * X_{n-1} + a * X_{n-2} \\ 1 * X_{n-1} + 0 * X_{n-2} \end{bmatrix} = \begin{bmatrix} X_n \\ X_{n-1} \end{bmatrix}$$

which is equal to our left hand side. □

Theorem 9.4. $\begin{bmatrix} X_n \\ X_{n-1} \end{bmatrix} = \begin{bmatrix} b & a \\ 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} X_2 \\ X_1 \end{bmatrix}$

Proof. Base Case $n = 2$: If $n = 2$, the left hand side is $\begin{bmatrix} X_2 \\ X_1 \end{bmatrix}$ and the right hand side is $\begin{bmatrix} b & a \\ 1 & 0 \end{bmatrix}^{0} \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} = \begin{bmatrix} X_2 \\ X_1 \end{bmatrix}$ So, the theorem holds when $n = 2$.

Inductive hypothesis: Suppose the theorem holds for some $n = k$, $k \geq 2$.

Inductive step: Let $n = k + 1$.
Then our left hand side is

$$\begin{bmatrix} X_{k+1} \\ X_k \end{bmatrix} = \begin{bmatrix} b & a \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X_k \\ X_{k-1} \end{bmatrix} = \begin{bmatrix} b & a \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b & a \\ 1 & 0 \end{bmatrix}^{k-2} \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} = \begin{bmatrix} b & a \\ 1 & 0 \end{bmatrix}^{k-2} \begin{bmatrix} X_2 \\ X_1 \end{bmatrix}$$

which is our right side. So, the theorem holds for $n = k + 1$. By the principle of mathematical induction, the theorem holds for all $n \in \mathbb{N}$. □

Appendix 1

Source Code for the Java Program

```
private void jButton1ActionPerformed(java.awt.event.ActionEvent evt) {
        // To calculate first n Fibonacci Numbers
        int n = Integer.parseInt(tf1.getText());
        int t1 = 1, t2 = 1;
        System.out.println("First " + n + " terms: ");

        for (int i = 1; i <= n; ++i) {
            System.out.print(t1 + "  ");
            int sum = t1 + t2;
            t1 = t2;
          t2 = sum;
        }
    }

    private void jButton2ActionPerformed(java.awt.event.ActionEvent evt) {
        // To calculate Z(m)
         int p = Integer.parseInt(tf2.getText());
         int p1 = 1, p2 = 1, num = 2, f = 1;
        while (f > 0)
        {
        f = (p1 + p2) % p;
            p1 = p2;
            p2 = f;
            num++;
        }
        ta1.append("First 0 for mod " + p + " = " + num + '\n');
    }

      private void jButton6ActionPerformed(java.awt.event.ActionEvent evt) {
        // To display the list of Z(m)
         int x = Integer.parseInt(tf2.getText());
           int n = Integer.parseInt(tf3.getText());
        for(int y=n;y>=x;x++){
         int p1 = 1, p2 = 1, num = 2, f = 1;
        while (f > 0){
            f = (p1 + p2) % x;
            p1 = p2;
            p2 = f;
            num++;
        }
        System.out.print("First 0 for mod " + x + " = " + num + '\n');}
    }

       private void jButton3ActionPerformed(java.awt.event.ActionEvent evt) {
        // To calculate the period F(mod m)
        int m = Integer.parseInt(tf2.getText());
         int a = 1, b = 1, c = 0;
           int  k = 2;
           c = (a+b)%m;
    while(b != 0){
        a = b;
        b = c;
        c = (a+b)%m;
        k++;
        }
        if (k%2 == 1){k = k*4;}
        else {
```

```
                if (c!=1){k = k*2;}
        }
        if (m == 2){k = 3;}
         ta2.append("period for mod" + m + " = " + k + '\n');
    }

    private void jButton4ActionPerformed(java.awt.event.ActionEvent evt) {
        //To display the list of period F(mod m)
         int m = Integer.parseInt(tf2.getText());
         int n = Integer.parseInt(tf3.getText());
        for(int y=n;y>=m;m++){
         int a = 1, b = 1, c = 0;
           int  k = 2;
          c = (a+b)%m;
  while(b != 0){
        a = b;
        b = c;
        c = (a+b)%m;
        k++;
        }
        if (k%2 == 1){k = k*4;}
        else {
                if (c!=1){k = k*2;}
        }
        if (m == 2){k = 3;}
System.out.print("period for mod" + m + " = " + k + '\n');}
  }
```